

# Palindromic Density

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## Abstract

In this paper we consider the palindromes that can be formed by taking unordered sets of  $n$  elements from an alphabet of  $b$  letters. In particular, we seek to find the probability that given a random member of this space we are able to re-arrange its elements to form a palindrome. We conclude by exploring the behaviour of this probability as  $n, b \rightarrow \infty$ .

## 1 Motivation

We begin with a short example to illustrate the problem. Suppose we have the set  $\{0, 1, \dots, 9\}$ . What unordered words of length 5 can we make? By unordered, we simply mean that ‘01156’ is the same word as ‘10651’. If we were to pick one of these words at random, what’s the probability that we could re-arrange the digits to create a palindrome?

Note that we assume every word to have equal probability of being picked, so this is *not* the same as asking “If we pick 5 random digits from 0-9, what are the chances we could re-arrange them to create a palindrome?”. In the second phrasing of the question, ‘01156’ is more likely than ‘00000’ as there are more sequences of ‘picks’ that will result in it.

The more intriguing problem arises when we generalise this, which we do in two ways. Rather than the digits 0-9, we instead have a set of  $b$  distinct characters, and rather than words of length 5 we will consider words of length  $n$ . In order for the problem to make sense, we will require  $n, b > 1$ .

## 2 Definitions and Notation

Before we start tackling this problem, we must first lay out some groundwork to formalise what we mean by this probability.

**Definition 2.1** (Multiset). A multiset is a set that allows repeated elements, for example  $\{1, 1, 2\}$ . In particular, and unlike regular sets, we have that  $\{1, 1, 2\} \neq \{1, 2\}$ . More formally, a multiset can be thought of as a set,  $A$ , paired with a counting function  $f : A \rightarrow \mathbb{N}$ .

**Definition 2.2** (Multiset Space). Let  $X_b$  be a set of  $b$  distinct elements, where  $b > 1$ . Then for  $n > 1$  we define the *multiset space* of cardinality  $n$  on  $X_b$  as:

$$X_b^n := \{X = \{x_1, \dots, x_n\} \mid x_j \in X_b, X \text{ is a multiset}\}$$

**Definition 2.3** (Palindromic).  $X \in X_b^n$  is called palindromic if:

$$\begin{aligned} \exists i_1, \dots, i_n \in \{1, \dots, n\} \quad i_k \neq i_l \quad \forall k \neq l \\ \text{s.t.} \quad (x_{i_1}, \dots, x_{i_n}) = (x_{i_n}, \dots, x_{i_1}) \end{aligned}$$

*Remark.* In simple terms, this means that the elements of the set can be arranged into a word that reads the same forwards or backwards. For example,  $\{1, 1, 2, 2, 3\}$  is palindromic (write it as 12321), whilst  $\{1, 1, 2, 3, 4\}$  is not.

**Definition 2.4** (Palindromic Density). Define  $P_b^n \subset X_b^n$  as:

$$P_b^n := \{X \in X_b^n \mid X \text{ is palindromic}\}$$

Then we define the *palindromic density* of  $X_b^n$  as:

$$PD(X_b^n) := \frac{|P_b^n|}{|X_b^n|}$$

In this paper we find a closed expression for  $PD(X_b^n)$  in terms of  $n$  and  $b$ .

### 3 Solution for $n = 5, b = 10$

With our problem precisely formulated, we can return to our initial example where  $n = 5$  and  $b = 10$ . It is difficult to compute the size of  $X_{10}^5$  due to the differing numbers of repetitions that can occur within its elements. A way of breaking down this problem is to partition  $X_{10}^5$  as follows:

$$X_{10}^5 = \bigcup_{1 \leq i \leq 7} Y_i$$

where:

$$\begin{aligned} Y_1 &:= \{\{a, b, c, d, e\} \in X_{10}^5 \mid a \neq b \neq c \neq d \neq e\} \\ Y_2 &:= \{\{a, a, b, c, d\} \in X_{10}^5 \mid a \neq b \neq c \neq d\} \\ Y_3 &:= \{\{a, a, b, b, c\} \in X_{10}^5 \mid a \neq b \neq c\} \\ Y_4 &:= \{\{a, a, a, b, c\} \in X_{10}^5 \mid a \neq b \neq c\} \\ Y_5 &:= \{\{a, a, a, b, b\} \in X_{10}^5 \mid a \neq b\} \\ Y_6 &:= \{\{a, a, a, a, b\} \in X_{10}^5 \mid a \neq b\} \\ Y_7 &:= \{\{a, a, a, a, a\} \in X_{10}^5\} \end{aligned}$$

In our partition, the palindromic sets are  $Y_3, Y_5, Y_6$  and  $Y_7$ . Computing the sizes of these sets is simple - for example,  $|Y_2| = \frac{10 \times 9 \times 8 \times 7}{3!} = 840$ . The first set is a slightly special case:  $|Y_1| = \binom{10}{5} = 252$ .

Thus:

$$PD(X_{10}^5) = \frac{|P_{10}^5|}{|X_{10}^5|} = \frac{|Y_3| + |Y_5| + |Y_6| + |Y_7|}{|Y_1| + |Y_2| + |Y_3| + |Y_4| + |Y_5| + |Y_6| + |Y_7|} = \frac{550}{2002}$$

## 4 The General Solution

We start by finding  $|X_b^n|$ , using an existing combinatorial result, with a proof adapted from [1].

**Theorem 4.1.**

$$|X_b^n| = \binom{n+b-1}{b-1}$$

*Proof.* Since  $b$  is finite, we can define an ordering on  $X_b$  and represent elements of  $X_b^n$  as  $\{x_1, x_2, \dots, x_n\}$  where  $x_i \preceq x_j \ \forall i < j$ .

Every element of  $X \in X_b^n$  can then be denoted uniquely as a series of ‘stars and bars’: we can lay out  $n$  ‘stars’ to represent the elements of the ordered multiset  $X$ , and then place  $(b-1)$  bars among them to separate out the distinct characters.

For example, the multiset  $\{0, 0, 1, 3, 4\} \in X_{10}^5$  would be represented as follows:

$$* * | * || * | * ||||$$

For every such arrangement, it is possible to get back to an element of  $X_b^n$ , and so our problem is reduced to finding the number of ways to arrange  $n$  stars and  $(b-1)$  bars in this way. Note that there are  $(n+b-1)$  positions in total, and we must choose  $(b-1)$  of these for our bars. We therefore have that the number of such representations is  $\binom{n+b-1}{b-1}$ , as required.  $\square$

**Theorem 4.2.** *If  $n$  is even, then*

$$|P_b^n| = |X_b^{\frac{n}{2}}|$$

*Proof.* We seek to construct a bijective function  $f : X_b^{\frac{n}{2}} \rightarrow P_b^n$ . In order for this to be well-defined, we will use the ordering of  $X_b$  discussed in the previous proof to represent elements in the domain uniquely.

Now construct a doubling function  $f : X_b^{\frac{n}{2}} \rightarrow X_b^n$  in the following way:

$$f(\{x_1, x_2, \dots, x_{\frac{n}{2}}\}) := \{x_1, x_1, x_2, x_2, \dots, x_{\frac{n}{2}}, x_{\frac{n}{2}}\}$$

Clearly  $f$  is injective. We can also verify that all sets produced by  $f$  are palindromic by arranging the elements as  $(x_1, x_2, \dots, x_{\frac{n}{2}}, x_{\frac{n}{2}}, \dots, x_2, x_1)$ . So  $f : X_b^{\frac{n}{2}} \rightarrow P_b^n$ . It remains to show that  $f$  is surjective.

Suppose  $Y \in P_b^n$ . Then by definition 2.3:

$$\begin{aligned} & \exists i_1, \dots, i_n \in \{1, \dots, n\} \quad i_k \neq i_l \quad \forall k \neq l \\ \text{s.t.} \quad & (y_{i_1}, \dots, y_{i_{\frac{n}{2}}}, y_{i_{\frac{n}{2}+1}}, \dots, y_{i_n}) = (y_{i_n}, \dots, y_{i_{\frac{n}{2}+1}}, y_{i_{\frac{n}{2}}}, \dots, y_{i_1}) \end{aligned}$$

In particular, this means we can write  $Y$  as  $\{y_{i_1}, y_{i_1}, y_{i_2}, y_{i_2}, \dots, y_{i_{\frac{n}{2}}}, y_{i_{\frac{n}{2}}}\}$ , so  $f(X) = Y$  where  $X := \{y_{i_1}, y_{i_2}, \dots, y_{i_{\frac{n}{2}}}\} \in X_b^{\frac{n}{2}}$

Hence  $f$  is bijective and the statement holds.  $\square$

**Theorem 4.3.** *If  $n$  is odd, then*

$$|P_b^n| = b|P_b^{n-1}|$$

*Proof.* From the proof of 4.2, we know that every element of  $P_b^{n-1}$  can be written as  $\{x_1, x_1, x_2, x_2, \dots, x_{\frac{n-1}{2}}, x_{\frac{n-1}{2}}\}$ . It is clear from this expression that we cannot change a single element of the set and preserve the palindromic property. We can therefore deduce that every element of  $P_b^{n-1}$  differs by at least two elements.

Suppose  $X \in P_b^{n-1}$ . Then if we add another element to it,  $x_* \in X_b$ , the resulting set must belong to  $P_b^n$ . This holds because we can arrange our new set as follows, showing that it is palindromic:

$$(x_1, x_2, \dots, x_{\frac{n-1}{2}}, x_*, x_{\frac{n-1}{2}}, \dots, x_2, x_1)$$

For every  $X \in P_b^{n-1}$ , it is possible to produce  $b$  elements of  $P_b^n$  in this way. These elements are uniquely determined by our choice of  $X$  and  $x_*$  - recall that elements of  $P_b^{n-1}$  differ by at least two elements. Hence

$$|P_b^n| \geq b|P_b^{n-1}|$$

We now show that every element of  $P_b^n$  can be made in this way. Suppose  $Y \in P_b^n$ . Then by definition 2.3:

$$\begin{aligned} & \exists i_1, \dots, i_n \in \{1, \dots, n\} \quad i_k \neq i_l \quad \forall k \neq l \\ \text{s.t.} \quad & (y_{i_1}, \dots, y_{i_{\frac{n-1}{2}}}, y_{i_{\frac{n+1}{2}}}, y_{i_{\frac{n+3}{2}}}, \dots, y_{i_n}) = (y_{i_n}, \dots, y_{i_{\frac{n+3}{2}}}, y_{i_{\frac{n+1}{2}}}, y_{i_{\frac{n-1}{2}}}, \dots, y_{i_1}) \end{aligned}$$

In particular, this means we can write  $Y$  as  $\{y_{i_1}, y_{i_1}, \dots, y_{i_{\frac{n-1}{2}}}, y_{i_{\frac{n-1}{2}}}, y_{i_{\frac{n+1}{2}}}\}$ . But this is exactly what we get by adding the element  $x_* := y_{i_{\frac{n+1}{2}}}$  to  $X := \{y_{i_1}, y_{i_1}, \dots, y_{i_{\frac{n-1}{2}}}, y_{i_{\frac{n-1}{2}}}\} \in P_b^{n-1}$ .

Hence  $|P_b^n| \leq b|P_b^{n-1}|$  and the statement holds.  $\square$

We now combine the above three theorems to arrive at the closed form we were looking for.

**Corollary 4.3.1.**

$$PD(X_b^n) = \begin{cases} \frac{\binom{\frac{n}{2}+b-1}{b-1}}{\binom{n+b-1}{b-1}} & \text{if } n \text{ is even} \\ b \frac{\binom{\frac{n-1}{2}+b-1}{b-1}}{\binom{n+b-1}{b-1}} & \text{if } n \text{ is odd} \end{cases}$$

## 5 Behaviour as $b, n \rightarrow \infty$

The result obtained in the previous section can be expanded and simplified into the following form, which has advantages when exploring convergence:

$$PD(X_b^n) = \begin{cases} \prod_{i=\frac{n+2}{2}}^n \frac{i}{i+b-1} & \text{if } n \text{ is even} \\ b \prod_{i=\frac{n+1}{2}}^n \frac{i}{i+b-1} & \text{if } n \text{ is odd} \end{cases}$$

First, we consider the case when  $b \rightarrow \infty$ . It makes sense that if we increase  $b$ , the probability of a set being palindromic should decrease. This is because with more distinct elements to choose from, a smaller proportion of words (or multisets) will contain the number of repetitions we need. The next result, then, shouldn't come as much of a surprise.

**Theorem 5.1.**

$$PD(X_b^n) \rightarrow 0 \quad \text{as } b \rightarrow \infty$$

*Proof.* We notice that  $i > 1$  for every term in the product. Hence, if  $n$  is even:

$$\prod_{i=\frac{n+2}{2}}^n \frac{i}{i+b-1} < \frac{n}{n+b-1} \rightarrow 0 \quad \text{as } b \rightarrow \infty$$

If  $n$  is odd, then we must use a slightly different bound:

$$\begin{aligned} b \prod_{i=\frac{n+1}{2}}^n \frac{i}{i+b-1} &< b \prod_{i=\frac{n+1}{2}}^n \frac{i}{b} \quad \text{as } i, b > 1 \quad \forall i \\ &< b \prod_{i=\frac{n+1}{2}}^n \frac{n}{b} \quad \text{as } n \geq i \quad \forall i \\ &= \frac{bn^{\frac{n+1}{2}}}{b^{\frac{n+1}{2}}} \\ &= \frac{n^{\frac{n+1}{2}}}{b^{\frac{n-1}{2}}} \rightarrow 0 \quad \text{as } b \rightarrow \infty \end{aligned}$$

□

Now consider the case when  $n \rightarrow \infty$ . At first glance, it feels intuitive that the density should go to zero again – by increasing the length of our words we're requiring more and more letters to pair off. However, notice that once the value of  $n$  exceeds that of  $b$ , we are *guaranteed* to start encountering duplicates - things are not quite as simple here. It is at least true that the density decreases with  $n$ , which we prove next.

**Theorem 5.2.** *Let  $k \in \mathbb{N}$ . Then the following results hold:*

$$\begin{aligned} \forall b > 1, PD(X_b^{2k}) \text{ is strictly decreasing in } k \\ \forall b > 2, PD(X_b^{2k+1}) \text{ is strictly decreasing in } k \end{aligned}$$

*Remark.* Notice that we specify  $b > 2$  for the odd case. Why is this? When  $b = 2$  and  $n$  is odd, we have to make an odd number of picks from  $\{0, 1\}$  to create a set of  $X_b^n$ . Doing this will *always* produce a palindromic set. It's easy to verify that plugging  $b = 2$  into the formula from Corollary 4.3.1 results in a density of 1 for any odd-valued  $n$ .

*Proof.* We start with the even case.

$$\begin{aligned} PD(X_b^{2k}) &= \frac{\binom{k+b-1}{b-1}}{\binom{2k+b-1}{b-1}} \\ &= \frac{(k+b-1)!}{(b-1)!(k)!} \times \frac{(b-1)!(2k)!}{(2k+b-1)!} \\ &= \frac{(k+b-1)!(2k)!}{(k)!(2k+b-1)!} \end{aligned}$$

Now consider the next term, which we get by incrementing  $k$  by 1:

$$\begin{aligned} PD(X_b^{2k+2}) &= \frac{(k+b)!(2k+2)!}{(k+1)!(2k+b+1)!} \\ &= \delta(k, b) PD(X_b^{2k}) \end{aligned}$$

where

$$\begin{aligned} \delta(k, b) &:= \frac{(k+b)(2k+1)(2k+2)}{(k+1)(2k+b)(2k+b+1)} \\ &= \frac{2(k+b)(2k+1)}{(2k+b)(2k+b+1)} \\ &= \frac{4k^2 + (4b+2)k + 2b}{4k^2 + (4b+2)k + b^2 + b} =: \frac{\alpha(k, b)}{\beta(k, b)} \end{aligned}$$



Subtracting the numerator from the denominator leaves:

$$\begin{aligned}\beta(k, b) - \alpha(k, b) &= b^2 + b - 2b \\ &= b(b - 1) > 1 \quad \forall b > 1\end{aligned}$$

Hence  $\delta(k, b) < 1 \quad \forall b > 1$  and  $PD(X_b^{2k})$  is strictly decreasing in  $k$ .

For the odd case, we can expand out  $PD(X_b^{2k+1})$  and  $PD(X_b^{2k+3})$  in the same way to obtain:

$$\delta(k, b) = \frac{4k^2 + (4b + 6)k + 6b}{4k^2 + (4b + 6)k + b^2 + 3b + 2} =: \frac{\alpha(k, b)}{\beta(k, b)}$$

Subtracting the numerator from the denominator again:

$$\begin{aligned}\beta(k, b) - \alpha(k, b) &= b^2 - 3b + 2 \\ &= (b - 2)(b - 1) > 1 \quad \forall b > 2\end{aligned}$$

Hence  $\delta(k, b) < 1 \quad \forall b > 2$  and  $PD(X_b^{2k+1})$  is strictly decreasing in  $k$ . □

**Theorem 5.3.**

$$PD(X_b^n) \rightarrow \begin{cases} \frac{1}{2^{b-1}} & n \text{ even}, n \rightarrow \infty \\ \frac{b}{2^{b-1}} & n \text{ odd}, n \rightarrow \infty \end{cases}$$

*Proof.* We first prove the result when  $n$  is even. Let  $n = 2k$ ,  $k \in \mathbb{N}$ . Then the number of terms in our product form is  $2k - (k + 1) + 1 = k$ . Since we're taking  $k \rightarrow \infty$ , we can set  $k > 2b$  and expand the first and last  $b$  terms as follows:

$$\begin{aligned}\prod_{i=k+1}^{2k} \frac{i}{i + b - 1} &= \frac{(k+1)}{\cancel{(k+b)}} \frac{(k+2)}{\cancel{(k+b+1)}} \cdots \frac{(k+b-1)}{\cancel{(k+2b-2)}} \frac{\cancel{(k+b)}}{\cancel{(k+2b-1)}} \\ &\quad \cdots \frac{\cancel{(2k-b+1)}}{\cancel{(2k)}} \frac{\cancel{(2k-b+2)}}{(2k+1)} \cdots \frac{\cancel{(2k-1)}}{(2k+b-2)} \frac{\cancel{(2k)}}{(2k+b-1)}\end{aligned}$$

After cancelling,  $(b - 1)$  terms remain on the top and bottom. Thus:

$$\begin{aligned}\prod_{i=k+1}^{2k} \frac{i}{i+b-1} &= \frac{(k+1)(k+2)\dots(k+b-1)}{(2k+1)(2k+2)\dots(2k+b-1)} \\ &= \frac{k^{b-1} + F(k)}{2^{b-1}k^{b-1} + G(k)}\end{aligned}$$

where  $F(k)$  and  $G(k)$  are polynomials in  $k$  of order  $(b - 2)$ . By L'Hôpital's Rule, it follows that:

$$\lim_{k \rightarrow \infty} \left( \prod_{i=k+1}^{2k} \frac{i}{i+b-1} \right) = \frac{1}{2^{b-1}}$$

The same argument applies when  $n$  is odd, noting additionally that

$$\lim_{k \rightarrow \infty} (bf(k)) = b \lim_{k \rightarrow \infty} (f(k))$$

□

## 6 Graphing $PD(X_b^n)$

The product form obtained in section 5 allows us to calculate the Palindromic Density for large values of  $n$  and  $b$  in order to produce surface graphs. These graphs demonstrate more visually the behaviour of  $PD(X_b^n)$  as  $n$  and  $b$  are increased.

Figure 1:  $PD(X_b^n)$  for  $n, b \in [2, 50]$

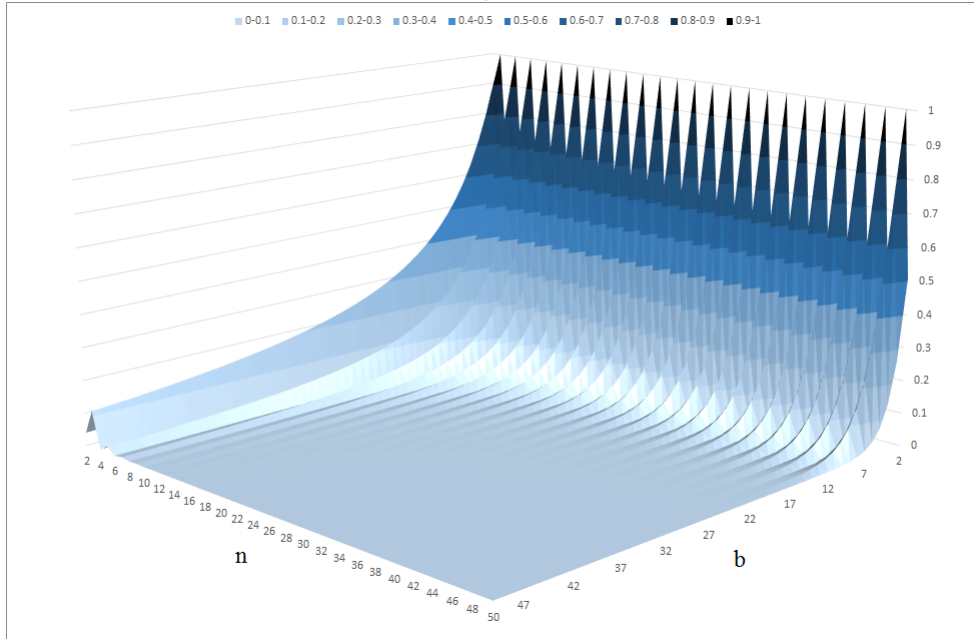


Figure 2:  $PD(X_b^n)$  for  $n, b \in [10, 50]$

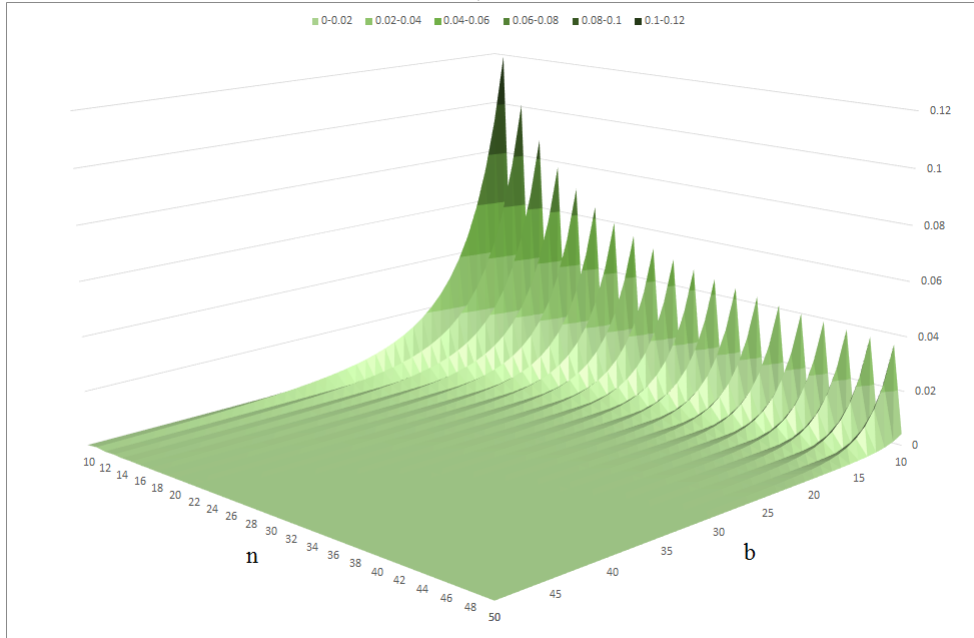
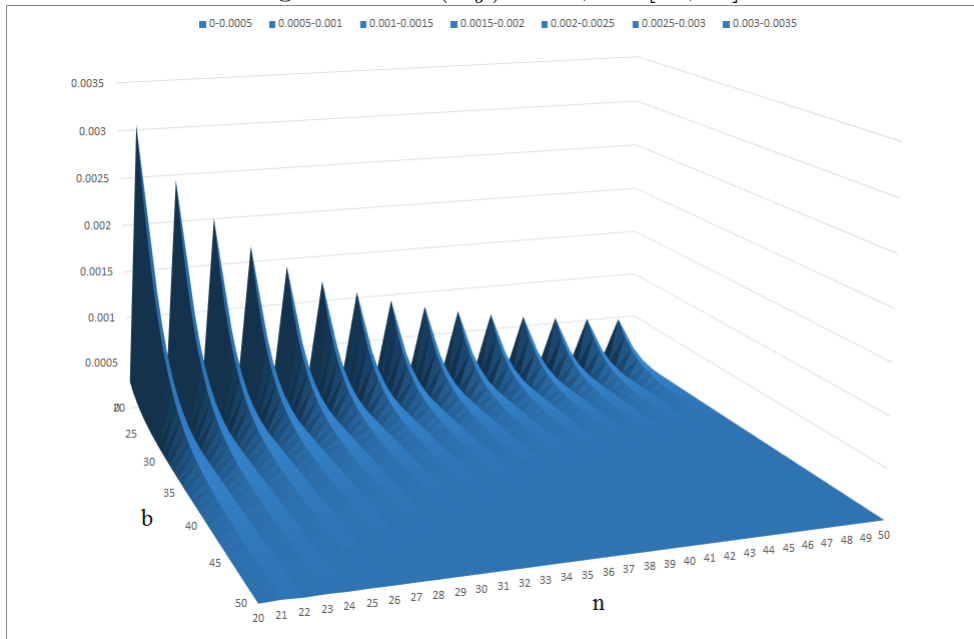


Figure 3:  $PD(X_b^n)$  for  $n, b \in [20, 50]$



## References

- [1] Edward A. Scheinerman. *Mathematics: A Discrete Introduction*. Cengage Learning, 105-106, 2012.